Collinear homographic orbits in the general problem of N bodies

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Sunto – Si ripropone uno studio dell'esistenza di soluzioni collineari del problema degli N corpi, come trattato da Eulero e Lagrange. Si segue la traccia proposta da Newton nei *Principia*, ossia considerare il problema nella sua generalità, senza assumere che la forza di attrazione dipenda dall'inverso del quadrato della distanza. Si mostra che in generale esistono orbite concentriche circolari, ovvero equilibri relativi. Per contro, si mostra che esistono orbite omografiche solo se le forze sono potenze della distanza.

Parole chiave – Problema dei tre corpi; Orbite stazionarie; Orbite omografiche.

Abstract – We revisit the problem of collinear solutions of the problem of N bodies, as investigated by Euler and Lagrange. Unlike most existing studies, we consider a general class of attractive forces. In this respect, we follow Newton's attitude in *Principia* of first considering the problem in its generality, without assuming that the force obeys the gravitational inverse square law. We find that circular concentric orbits, also named relative equilibria, exist as a general fact. Conversely, we show that homographic orbits do exist only for forces that obey a power law.

Keywords – Problem of three bodies; Stationary orbits; Homographic orbits.

Introduction and statement of results

The aim of this paper is to revisit the study of collinear solutions of the problem of N bodies interacting through general force fields, not restricted to the case of Newton's gravitational force.

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Collinear solutions of the problem of three bodies have been investigated by Euler (Euler 1766; Euler 1767; Euler 1770). He openly recognized that a complete study of the problem of three bodies was inaccessible to his methods of Analysis. Therefore, Euler decided to begin with the simplified case of motion on a line. This included, in particular, a possible motion of the Moon on a state of perpetual alignment on syzygy. The first wide-scale study is due to Lagrange (Lagrange 1772), who succeeded in writing equations for the mutual distances among the bodies. He focused on the search of solutions such that the mutual distances, or at least the ratios between any two of them, remain constant. This is indeed the class of solution called *relative equilibria* and *homographic orbits*, respectively. In addition to Euler's collinear solutions, Lagrange also discovered the *triangular solutions*: the bodies are located at the vertices of an equilateral triangle, which revolves in space. A generalization of the collinear solutions of Euler and Lagrange has been found by Forest Ray Moulton (Moulton 1910). An extension to the case of four bodies had been previously published by Rudolf Lehmann-Filhés (Lehmann–Filhés 1891).

Our interest in collinear solution is connected with paper (Giorgilli and Guicciardini 2021). As discussed there, Newton made an accurate study before making the conclusion that the gravitational force obeys the inverse square law, with particular attention to forces obeying a power law. However, the great majority of papers on collinear, or planar stationary, or homographic orbits, consider only Newton's gravitational law. Thus, if only for pure academic interest, we take the attitude of considering a more general class of forces. As a first step, we focus our attention on collinear orbits, as Euler did. However, a similar investigation may be extended also to generalizations of Lagrange's triangular solutions, for instance polygonal configurations, that have been studied, for instance, in (Wintner 1941), and then in many papers. An extensive exposition concerning recent works, with many references, may be found in (Moeckel 2014; Saari 2005; Saari 2011).

Let us come to the statement of our results. We assume that the force function $\varphi(r)$ satisfies the following hypotheses:

(H1) The intensity of the force is written as $mm'\varphi(r)$, where $r > 0$ is the distance between any two bodies and $m,\ m'$ are the masses.

(H2) The function $\varphi(r)$ is assumed to be positive, continuous and monotonically decreasing for increasing r, with $\varphi(r) \underset{r\to+\infty}{\longrightarrow} 0$. Both cases, $\varphi(0)$ finite or $\varphi(0) = +\infty$, are allowed.

(H3) The function $\varphi(r)$ is a convex function; that is, for $r < r'$ and for $0 \leq \lambda \leq 1$ we have $\varphi\big((1-\lambda)r + \lambda r'\big) \leq (1-\lambda)\varphi(r) + \lambda\varphi(r')$.

It is an easy matter to prove that hypotheses (H2) and (H3) entail that for any $\delta > 0$ and for $r < r'$ we have $\varphi(r) - \varphi(r+\delta) > \varphi(r') - \varphi(r'+\delta)$. Later, we shall need a more stringent assumption.

(H4) The force law is an inverse power of the distance, thus homogeneous; that is, $\varphi(r) = r^{-\alpha}$ with $0 < \alpha < 3$.

Here, the limits on the exponent α select a class forces which possess stable circular orbits, as is usual in discussing the problem of central forces. We note that our arguments apply to any negative power, but the cases selected here are the physically interesting ones.

We prove the following two theorems. First, using hypotheses (H1)–(H3), we prove the existence of relative equilibria.

Theorem 1. *Let N masses $m_1, \ldots, m_N$ be given, interacting with an attractive force $\varphi(r)$, where $r > 0$ is the distance between two bodies. Assume that $\varphi(r)$ satisfies the hypotheses (H1)–(H3) just stated. Then there exist $N!/2$ collinear configurations of relative equilibrium, with the masses aligned on a straight line which revolves with uniform angular velocity around the barycentre.*

The second theorem is concerned with the existence of homographic orbits corresponding to the collinear equilibria. This is a more intriguing problem: it requires some extra conditions on the forces.

Theorem 2. *With the hypotheses of Theorem 1, let also hypothesis (H4) be satisfied, namely $\varphi(r) = r^{-\alpha}$ with $0 < \alpha < 3$. Then, corresponding to every collinear solution, there exists a family of homographic orbits, such that the bodies remain aligned on a straight line revolving (non uniformly) around the barycentre, and the ratio of the relative distances between any two bodies remains constant.*

The curious fact is that we should restrict the force to depend on a power of the distance. This is indeed the condition found by Newton, when he was answering the following two questions for the problem of central motion; see (Giorgilli and Guicciardini 2021) for a transcription in our current mathematical language.

(i) *To determine the motion of the apsidal line for orbits very close to a circular one ((Newton 1686), Liber I, sectio IX, problema XXXI).*

(ii) *To find attractive forces such that the precession angle of the apsidal line does not depend on the radius of the circular orbit.*

Newton answers the first question for generic forces. Then he proves that the second question possesses a positive answer in case the force obeys a power law. Perhaps surprisingly, we need the same assumption.

1. Homographic Orbits, Central Configurations

The problem of homographic orbits is enunciated as follows:

To find orbits such that in a barycentric frame the geometric configuration of the bodies remains similar to itself during the motion.

1.1. The equations in the planar case

We choose an absolute rectangular frame with origin in the barycentre. The coordinates $\mathbf{x}_1, \ldots, \mathbf{x}_N$ of the bodies obey Newton's equations

$$m_j \ddot{\mathbf{x}}_j = -\sum_{k \neq j} m_j m_k \varphi(r_{j,k}) \frac{\mathbf{x}_j - \mathbf{x}_k}{r_{j,k}} , \quad j = 1, \ldots, N , \tag{1}$$

where $r_{j,k} = \|\mathbf{x}_j - \mathbf{x}_k\|$. The equations possess the classical first integrals of the barycentre, total momentum and total angular momentum:

$$\sum_{j=1}^{N} m_j \mathbf{x}_j = 0 , \quad \sum_{j=1}^{N} m_j \dot{\mathbf{x}}_j = 0 , \quad \sum_{j=1}^{N} \mathbf{x}_j \wedge m_j \dot{\mathbf{x}}_j = \Gamma . \tag{2}$$

Now we assume that the orbits lie on a fixed plane orthogonal to Γ, so that in a rectangular frame with the z axis parallel to Γ we have $z_j = 0$. Let $\xi_1, \ldots, \xi_N$ be N fixed vectors in the orbital plane.

Homographic orbits are represented as

$$\mathbf{x}_j = \varrho \, \mathsf{R}_\theta \xi_j , \quad \mathsf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

where $\varrho(t)$, $\theta(t)$ depend on time. We distinguish three cases.

(i) A *relative equilibrium* occurs if $\varrho(t) = \varrho_0$ remains constant. In this case the bodies revolve on circles around the barycentre, all with the same angular velocity $\dot{\theta} = \omega$. In a revolving frame with angular velocity ω the bodies remain fixed.

(ii) A *homothetic orbit* occurs when $\theta(t) = \theta_0$ remains constant. In this case the bodies move on rectilinear orbits; the geometric configuration is subject only to resizing with a factor $\varrho(t)$.

(iii) A *homographic orbit* occurs when neither $\varrho(t)$ nor $\theta(t)$ are constant. The geometric configuration is such that the ratio of the relative distance between any two bodies remains constant.

Thus, relative equilibria and homothetic orbits are particular cases of homographic orbits. In any case, we stress that the vectors $\xi_1,\ldots,\xi_N$ and the functions $\varrho(t)$, $\theta(t)$ are not arbitrary. Our aim is to prove that there exist solutions of Newton's equations (1).

Lemma 3. *Equations (1) for the orbits (3) are written as*

$$\frac{d}{dt}\varrho^2\dot\theta = 0\,,\quad \dot\theta = \frac{L}{\varrho^2}\,,\tag{4}$$

where L is a constant, and, denoting $r_{j,k} = \|\xi_j - \xi_k\|$,

$$\left(\ddot\varrho - \frac{L^2}{\varrho^3}\right)\xi_j = -\sum_{k\neq j} m_k\varphi(r_{j,k}\varrho)\frac{\xi_j-\xi_k}{r_{j,k}}\,,\quad j=1,\ldots,N\,.\tag{5}$$

The proof of the lemma reduces to writing Newton's equations in planar polar coordinates. This is a standard calculation in books on Mechanics.

1.2. Planar Central Configurations

Central Configurations are defined as follows (see (Wintner 1941), § 355).

Definition 4. *A system of bodies in space is said to be in a central configuration in case the total force acting on each body is proportional both to its mass and to its barycentric position.*

We emphasize that the enunciate is concerned only with the *configuration* of the system, that is, with the position of the bodies at a given fixed time, irrespective of the velocities.

In a central configuration the coordinates of the bodies must satisfy

$$\Psi\xi_j = \sum_{k\neq j} m_k\varphi(r_{j,k})\frac{\xi_j-\xi_k}{r_{j,k}}\,,\quad j=1,\ldots,N\,.\tag{6}$$

where $\Psi > 0$ is a constant. There is some freedom here.

Lemma 5. *Let $\xi_1,\ldots\xi_N$ be a solution of equations (6) with a given Ψ. Then the following holds true:*

(i) The vectors $\mathsf{R}_\alpha\xi_1,\ldots,\mathsf{R}_\alpha\xi_n$, where R_α is a plane rotation matrix with an arbitrary angle α, solve the equations with the same Ψ.

(ii) Given any $\sigma > 0$, let us rescale $\xi_1 = \sigma\xi'_1, \ldots, \xi_N = \sigma\xi'_N$, so that the relation $r_{j,k} = \sigma r'_{j,k}$ applies to the corresponding distances. Then the equations are changed as

$$\sigma\Psi\,\xi'_j = \sum_{k\neq j} m_k\varphi(\sigma r'_{j,k})\frac{\xi'_j - \xi'_k}{r'_{j,k}}\ , \qquad \left(\frac{\xi'_j - \xi'_k}{r'_{j,k}} = \frac{\xi_j - \xi_k}{r_{j,k}}\right)$$

The proof is straightforward: just make the necessary substitutions.

Proposition 6. *Let the bodies move on homographic orbits, including also the subcases of relative equilibria and of homothetic orbits. Then at any instant of time the bodies are in a central configuration.*

The claim is self-evident in view of Equation (5).

Corollary 7. *The quantities Ψ and L in Lemma 3, and the angular velocity ω are subject to the following rules.*

(i) For a relative equilibrium, we have $L^2/\varrho_0^3 = \Psi \neq 0$, with ϱ_0 the fixed initial value, and the angular velocity is $\omega = L/\varrho_0^2$.

(ii) For a homothetic orbit, we have $\omega = L = 0$; Ψ is replaced by a positive function $\psi(\varrho)$ as in the next case (iii).

(iii) For a homographic orbit the constant L is determined by the initial conditions; $\omega(t) = \dot\theta$ obeys Equation (4); Ψ is replaced by a positive function $\psi(\varrho)$ satisfying

$$\psi(\varrho)\xi_j = \sum_{k\neq j} m_k\varphi(\varrho r_{j,k})\frac{\xi_j - \xi_k}{r_{j,k}}\ , \quad j = 1,\ldots,N\ , \tag{7}$$

where $\xi_1, \ldots \xi_N$ are constant vectors.

Proof. (i) Equation (5) is rewritten as

$$\left(\ddot\varrho - \frac{L^2}{\varrho^3}\right) = -\Psi\ .$$

A solution of relative equilibrium exists only if $\ddot\varrho = \dot\varrho = 0$; if so, then we have $L^2/\varrho_0^3 = \Psi$, and $\dot\theta = \omega = \sqrt{\Psi/\varrho_0}$ in view of Equations (4).
(ii) Equation (4) entails $L = 0$.
(iii) The distances $r_{j,k}$ in Equation (5) are rescaled by a factor $\varrho(t)$. Hence, the right member obviously depends on ϱ. Since in Equation (3) the vectors ξ must be fixed, Ψ needs to depend on ϱ, but not on ξ. Q.E.D.

2. Collinear central configurations

In the case of collinear orbits, the vectors $\xi_1, \ldots, \xi_N$ must all be parallel; hence a single coordinate describes the position of the bodies on a line, and the angle θ describes how the line is rotated in the plane.

Let us denote by $\xi = (\xi_1, \ldots, \xi_N)$, ordered as $\xi_1 < \ldots < \xi_N$, the coordinates of the N bodies on the line, taking the barycentre as the origin. The equation of the barycentre gives $m_1\xi_1 + \ldots + m_N\xi_N = 0$.

In view of Lemma 5, we may suppose that we have $\varrho = 1$ at a given instant, so that the configuration of the bodies is given by ξ (on need, just rescale ξ). Denote by $r_{j,k} = |\xi_j - \xi_k|$ the relative distances between two bodies. The attractive force exerted by the k-th body on the j-th one is $\pm m_j m_k \varphi(r_{j,k})$, where the sign is positive if $j < k$, negative otherwise. Hence, $\xi_1, \ldots, \xi_n$ must obey the system of equations (forgetting the common factor m_j and assigning a positive sign to the attractive force)

$$\Psi\xi_j = \sum_{k=1}^{j-1} m_k\varphi(r_{k,j}) - \sum_{k=j+1}^{N} m_k\varphi(r_{j,k}) , \quad j = 1, \ldots, N , \tag{8}$$

where Ψ is a constant factor which controls the proportion between the attractive force acting on a body and the distance from the barycentre. Here, the first and the last sum are empty for $j = 1$ and $j = N$, respectively. These equations are not independent, since they must satisfy the equation of the barycentre. We introduce the $n = N - 1$ distances between two neighboring bodies

$$\delta_j = r_{j,j+1} = \xi_{j+1} - \xi_j , \quad j = 1, \ldots, n , \tag{9}$$

so that for $j < k$ we have $r_{j,k} = \xi_k - \xi_j = \delta_j + \ldots + \delta_{k-1}$. We will also use the collective notation $\delta = (\delta_1, \ldots, \delta_n)$. Note that by definition we have $\delta_j > 0$, in view of the ordering of the coordinates ξ; this excludes the case of two or more bodies occupying the same position. We subtract the j-th equation from the next one; thus defining

$$\begin{aligned} Q_j(\delta_j) &= \Psi\delta_j - (m_j + m_{j+1})\varphi(\delta_j) , \\ G_j(\delta) &= -\sum_{k=1}^{j-1} m_k\big[\varphi(r_{k,j}) - \varphi(r_{k,j} + \delta_j)\big] \\ &\quad - \sum_{k=j+2}^{n+1} m_k\big[\varphi(r_{j+1,k}) - \varphi(\delta_j + r_{j+1,k})\big] . \end{aligned} \tag{10}$$

We recall that the first and the last sum in the definition of $G_j(\delta)$ are empty for $j = 1$ and $j = n + 1 = N$, respectively. With this setting, we write the system of n equations

$$Q_j(\delta_j) = G_j(\delta)\ , \quad j = 1, \ldots, n\ . \tag{11}$$

Denoting by $\ell = r_{1,n+1} = \sum_{j=1}^{n} \delta_j$ the length of the chain of bodies, we add up all equations (11), and get the useful equation

$$\hat{Q}(\ell) = \hat{G}(\delta)\ , \tag{12}$$

where

$$\begin{aligned} \hat{Q}(\ell) &= \Psi\ell - (m_1 + m_{n+1})\varphi(\ell)\ ,\\ \hat{G}(\delta) &= \sum_{k=2}^{n} m_k\big[\varphi(r_{k,n+1}) + \varphi(r_{1,k})\big]\ . \end{aligned} \tag{13}$$

This establishes the relation between the free parameter $\Psi > 0$ and the length ℓ of the chain of bodies, which is thus determined once the distances δ are found. We show how to find Equation (13). Exploiting the expression for $G_j(\delta)$ in (10), and appropriately removing the meaningless terms in $G_1(\delta)$ and $G_n(\delta)$, we calculate the sum $\sum_{j=1}^{n} G_j(\delta)$ as

$$\begin{aligned} &-\sum_{j=2}^{n}\sum_{k=1}^{j-1} m_k\big[\varphi(r_{k,j}) - \varphi(r_{k,j+1})\big] - \sum_{j=1}^{n-1}\sum_{k=j+2}^{n+1} m_k\big[\varphi(r_{j+1,k}) - \varphi(r_{j,k})\big]\\ &= -\sum_{k=1}^{n-1} m_k \sum_{j=k+1}^{n}\big[\varphi(r_{k,j}) - \varphi(r_{k,j+1})\big] - \sum_{k=3}^{n+1} m_k \sum_{j=1}^{k-2}\big[\varphi(r_{j+1,k}) - \varphi(r_{j,k})\big]\\ &= -\sum_{k=1}^{n-1} m_k\big[\varphi(r_{k,k+1}) - \varphi(r_{k,n+1})\big] - \sum_{k=3}^{n+1} m_k\big[\varphi(r_{k-1,k}) - \varphi(r_{1,k})\big]\ . \end{aligned}$$

Here, in the intermediate line the sums have been exchanged, with an appropriate resetting of the limits. This produces two telescopic sums over the index j, which result in the simplified expression of the third line. Using the latter expression, we add up Equations (11), and using also $r_{k,k+1} = \delta_k$ we compensate the terms $(m_j + m_{j+1})\varphi(\delta_j)$ on the left side; thus, with some recasting, we get Equation (12).

With this setting, the problem of finding a central equilibrium is enunciated as follows:

Given $\Psi > 0$*, to find a solution of Equations (11) and (10).*

That is, we are actually looking for a family of central configurations parameterized by the positive quantity Ψ.

Proposition 8. *With the hypotheses of Theorem 1 on the force* $\varphi(r)$*, for any finite* $\Psi > 0$*, and for any set of positive masses* $m_1, \ldots, m_N$*, the system of Equations (11) possesses a unique solution.*

The rest of the present section is devoted to the proof.

2.1. Preliminary general facts

By definition, the variables δ are all positive; hence we may consider them as belonging to the positive quadrant $\mathbb{R}^n_+$ of $\mathbb{R}^n$.

We denote by Δ_j the set of the $n-1$ variables δ_k with $k \neq j$, namely

$$\Delta_j = \{\delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_n\}.$$

When we say that Δ_j is fixed, we mean that every variable in Δ_j is fixed, so that the sole variable δ_j may vary.

Lemma 9. *The following properties hold true.*

(i) The functions $Q_j(\delta_j)$ *and* $\hat{Q}(\ell)$ *are continuous and monotonically increasing between the limits*

$$\lim_{\delta_j \to 0} Q_j(\delta_j) = -(m_j + m_{j+1})\varphi(0) \ , \qquad \lim_{\delta_j \to +\infty} Q_j(\delta_j) = +\infty \ ,$$

and

$$\lim_{\ell \to 0} \hat{Q}(\ell) = -(m_1 + m_{n+1})\varphi(0) \ , \qquad \lim_{\ell \to +\infty} \hat{Q}(\ell) = +\infty \ ,$$

where $\varphi(0)$ *may be either finite or infinite, depending on the character of the function* $\varphi(r)$*.*

(ii) Let Δ_j *be fixed, so that* $G_j(\delta)$ *may be considered as depending only on* δ_j*. This function is negative, continuous and monotonically decreasing between the limits*

$$\lim_{\delta_j \to 0} G_j(\delta) = 0 \ ,$$

$$\lim_{\delta_j \to +\infty} G_j(\delta) = -\sum_{k=1}^{j-1} m_k \big[\varphi(r_{k,j})\big] - \sum_{k=j+2}^{N} m_k \big[\varphi(r_{j+1,k})\big] \ ,$$

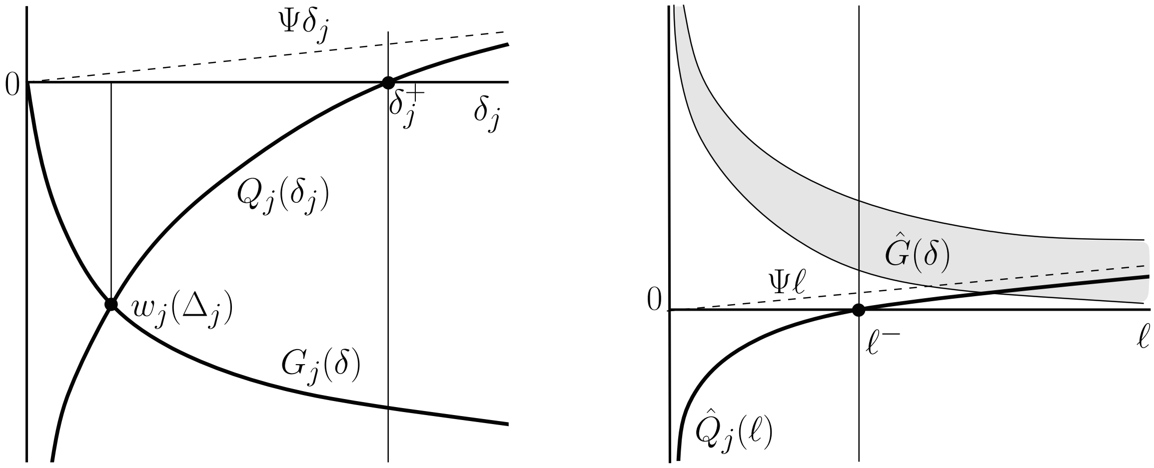

Fig. 1 – Illustrating the argument of proof of Lemmas 10 and 11. Left: the functions $Q_j(\delta_j)$ and $G_j(\delta)$, with Δ_j fixed, with the intersection $w_j(\Delta_j)$ and the upper bound δ_j^+. Right: the function $\hat{Q}(\ell)$ and the lower bound ℓ^-; $\hat{G}(\delta)$ is positive, and its graph lies in the gray band.

(iii) For $k \neq j$, let Δ_k be fixed. Then $G_j(\delta)$, considered as depending only on δ_k, is a continuous and monotonically increasing function.

(iv) The function $\hat{G}(\delta)$ is positive, continuous and monotonically decreasing between a positive, possibly infinite value for $\ell \to 0$, and with $\hat{G}(\delta) \to 0$ if all distances $r_{j,k}$ tend to $+\infty$. Moreover, it monotonically decreases if any of the distances $r_{j,k}$ increases.

Proof. According to Formulæ (10) and (13), all functions considered here are sums of continuous functions; hence they are continuous. In view of our hypotheses on $\varphi(r)$, for any positive h we have

$$\varphi(h) - \varphi(h+\delta_j) \quad \begin{cases} = 0 & \text{for} \quad \delta_j \to 0\ , \\ = \varphi(h) & \text{for} \quad \delta_j \to +\infty\ . \end{cases} \tag{14}$$

(i) The function $\varphi(r)$ is positive and monotonically decreasing, by hypothesis. The limits in the statement are straightforward consequences of Equation (14).

(ii) Every pair of square brackets in the expression (10) of $G_j(\delta)$ contains a difference of the form $\varphi(h) - \varphi(h+\delta_j)$, where $h > 0$ is one of the distances $r_{k,j}$ with $k < j$ or $r_{j+1,k}$ with $j+1 < k$, independent of δ_j; since Δ_j is fixed, h is fixed, too. Then the difference is a monotonically increasing continuous function of δ_j, which enters with negative sign in $G_j(\delta)$. The limits are a straightforward consequence of Equations (14).

(iii) Every square bracket in the expression of the function G_j contains a difference of either form

$$\varphi(h+\delta_j)-\varphi(h) \quad \text{or} \quad \varphi(h+\delta_k+\delta_j)-\varphi(h+\delta_k)\ ,$$

where $h>0$ is a finite sum of some quantities δ, excluding both δ_j and δ_k. In the first case the difference does not depend on δ_k, and does not affect G_j. In the second case, let $\delta_k<\delta'_k$. Then, in view of the convexity hypothesis (H3) on the force $\varphi(r)$, we have

$$\varphi(h+\delta_k)-\varphi(h+\delta_k+\delta_j)>\varphi(h+\delta'_k)-\varphi(h+\delta'_k+\delta_j)\ .$$

This enters with negative sign in $G_j(\delta)$, which proves the claim.
(iv) The function $\hat{G}(\delta)$ is a sum of positive functions $\varphi(r_{jk})$, which are all positive, continuous and monotonically decreasing; hence it inherits these properties. For $\ell\to 0$ we have $\varphi(r_{j,k})\to\varphi(0)$ for every j,k; this gives a positive, possibly infinite sum. If $r_{j,k}\to+\infty$ then $\varphi(r_{j,k})\to 0$, which proves the second limit. Q.E.D.

In Fig. 1 we represent the qualitative graphs of the functions $Q_j(\delta_j)$ and $G_j(\Delta_j)$ (left) and of $\hat{Q}(\ell)$ (right). The function $\hat{G}(\delta)$ is positive, which is enough in order to establish some bounds on the solution of Equation (11).

Lemma 10. *For $\Psi>0$ finite, the following holds true.*

(i) There are positive numbers $\ell^-<\ell^+$ such that the length ℓ of the chain of bodies satisfies

$$\ell^-<\ell<\ell^+\ .$$

(ii) For $j=1,\ldots,n$, there are positive numbers $\delta_j^-<\delta_j^+$ such that a solution of Equations (11), if it exists, must belong to the n-dimensional rectangle

$$\Pi=(\delta_1^-,\delta_1^+)\times\ldots\times(\delta_n^-,\delta_n^+)\ .$$

Proof. The right member of Equation (12) is a positive quantity. This, in view of Lemma 9-(i), entails that the equation $\hat{Q}(\ell)=0$ possesses a single solution ℓ^- which provides the lower bound $\ell^-<\ell$. Since $G_j(\delta)$ is negative, the solution δ_j^+ of the approximated equation $Q_j(\delta_j)=0$ provides the bound $\delta_j<\delta_j^+$. In turn, this provides also the upper bound $\ell<\ell^+=\sum_{j=1}^{n}\delta_j^+$. Coming to the existence of lower bounds δ_j^-, the case $\varphi(0)$ finite is trivial; hence we examine the case $\varphi(0)=+\infty$. Recall that $\sum_j\delta_j\geq\ell^->0$, so that at least one δ_j must be positive. Let us isolate two relevant terms in the second Equation (10), so as to rewrite it in the convenient form

(15)
$$\begin{aligned}G_j(\delta) = &-m_{j-1}\bigl[\varphi(\delta_{j-1})-\varphi(\delta_{j-1}+\delta_j)\bigr] - m_{j+2}\bigl[\varphi(\delta_{j+1})-\varphi(\delta_j+\delta_{j+1})\bigr]\\ &-\sum_{k=1}^{j-2} m_k\bigl[\varphi(r_{k,j}) - \varphi(r_{k,j}+\delta_j)\bigr]\\ &-\sum_{k=j+3}^{n+1} m_k\bigl[\varphi(r_{j+1,k}) - \varphi(\delta_j + r_{j+1,k})\bigr]\ .\end{aligned}$$

In view of $\varphi(r)$ being convex, in the two sums we have, respectively,

$$\begin{aligned}&\varphi(\delta_{j-1}) - \varphi(\delta_{j-1}+\delta_j) > \varphi(r_{k,j}) - \varphi(r_{k,j}+\delta_j)\ ,\\ &\varphi(\delta_{j+1}) - \varphi(\delta_j+\delta_{j+1}) > \varphi(r_{j+1,k}) - \varphi(\delta_j + r_{j+1,k})\ .\end{aligned}$$

Putting these inequalities in Equations (15) and introducing the abbreviated notations $\mu_1 = \nu_n = 0$, and

$$\begin{aligned}\mu_j &= m_1 + \ldots + m_{j-1} &&\text{for}\quad j > 1\ ,\\ \nu_j &= m_{j+2} + \ldots + m_{n+1} &&\text{for}\quad j < n\ .\end{aligned}$$

we find that for any δ we have

(16)
$$G_j(\delta) > -\mu_j\bigl[\varphi(\delta_{j-1})-\varphi(\delta_{j-1}+\delta_j)\bigr] - \nu_j\bigl[\varphi(\delta_{j+1})-\varphi(\delta_j+\delta_{j+1})\bigr]\ .$$

Let us now consider the modified system of equations, for $j = 1,\ldots,n$,

(17)
$$\begin{aligned}\Psi\delta_j - (m_j + m_{j+1})\varphi(\delta_j) = &-\mu_j\bigl[\varphi(\delta_{j-1}) - \varphi(\delta_{j-1}+\delta_j)\bigr]\\ &-\nu_j\bigl[\varphi(\delta_{j+1}) - \varphi(\delta_j+\delta_{j+1})\bigr]\ ;\end{aligned}$$

this is obtained from Equations (11), replacing $G_j(\delta)$ with the right members of (16). Therefore, a solution δ' of these equations, compared with the corresponding solution δ^* of Equations (11), must satisfy $\delta'_j < \delta^*_j$. It is immediate to see that it must be $\delta'_j > 0$, in view of the following argument. For $j = 1$, the equation is written as

$$\Psi\delta_1 - (m_1 + m_2)\varphi(\delta_1) = -\nu_1\bigl[\varphi(\delta_2) - \varphi(\delta_1+\delta_2)\bigr]\ .$$

For $\delta_1 \to 0$, the left member tends to $-\infty$, while the right member tends to zero. We may imagine that we should let also $\delta_2 \to 0$, but this is the physically meaningless solution. Moreover, the limit is ill defined.[1] The same objection applies to the cases $j > 1$, as the reader may easily see. This entails that all lower bounds δ_j^- must be positive. Q.E.D.

We may find explicit values for δ^- trough Equation (17); for instance, trying $\delta_j^- = \ell^-/n$. Define the functions $G_j^-(\delta_j)$ of the sole variable δ_j as

$$(18)\quad G_j^-(\delta_j) = -\mu_j\bigl[\varphi(\delta_{j-1}^-)-\varphi(\delta_{j-1}^-+\delta_j)\bigr]-\nu_j\bigl[\varphi(\delta_{j+1}^-)-\varphi(\delta_j)+\delta_{j+1}^-\bigr]\,.$$

For $\delta_j > \delta_j^-$ we have $G_j(\delta) > G_j^-(\delta_j)$, wich is a stronger inequality than (16). Thus, the claim (ii) of Lemma 10 holds true with the selected values of δ^-.

Lemma 11. *For any $j = 1,\ldots,n$, Equation (11) determines a family of surfaces*

$$\Sigma_j = \bigl\{\delta \in \mathbb{R}_+^n : Q_j(\delta_j) = G_j(\delta)\bigr\}\,,\quad j = 1,\ldots,n\,.$$

The surface Σ_j is uniquely projected on the positive quadrant of the variables Δ_j, and is the graph of a continuous and one valued function $\delta_j = w_j(\Delta_j)$, with image in the interval $0 < \delta_j < \delta_j^+$.

Proof. We use Bolzano's theorem. By Lemma 9, for Δ_j fixed, the graphs of the functions $Q_j(\delta_j)$ and $G_j(\delta)$ possess an intersection at a single point δ_j (see Fig. 1). Letting Δ_j to vary, we get the function $w_j(\Delta_j)$, which by construction is continuous and one-valued, and defines the surface Σ_j. The bound $w_j(\Delta_j) < \delta_j^+$ is stated in Lemma 10. Q.E.D.

Corollary 12. *Let $k \neq j$, and Δ_k be fixed. Then $w_j(\Delta_j)$ is continuous, one valued and monotonically increasing with δ_k. Let $\delta_k^- \le \delta_k \le \delta_k^+$. Then the curve $\Sigma_j \cap \Pi$ joins continuously the points $\bigl(\delta_k^-, w_j(\delta_k^-)\bigr)$ and $\bigl(\delta_k^+, w_j(\delta_k^+)\bigr)$, and we have*

$$(19)\qquad \delta_j^- < w_j(\Delta_j)\Big|_{\delta_k=\delta_k^-} < \delta_j^+\,,\quad \delta_j^- < w_j(\Delta_j)\Big|_{\delta_k=\delta_k^+} < \delta_j^+\,.$$

Proof. Let $\delta_k < \delta_k'$, and the corresponding Δ_j and Δ_j' differ only because δ_k is replaced by δ_k'. For $\delta_j = w_j(\Delta_j)$ fixed the function $Q_j(\delta_j)$ does not change. Conversely, by Lemma 9–(iii), $G_j(\delta)$ increases with $\delta_k \to \delta_k'$. Hence, the unique solution of the equation $Q_j(\delta_j) = G_j(\delta)$ is such that

[1] This point will be discussed in some detail for the case $n = 2$, in Section 2.2.

$\delta_j' = w_j(\Delta_j') > w_j(\Delta_j)$ (see Fig. 1). Let now $\delta_k \in [\delta_k^-, \delta_k^+]$. The intersection $\Sigma_j \cap \Pi$, with Δ_k fixed, is a piece of continuous, monotonically increasing curve, joining the points $\bigl(\delta_k^-, w_j(\delta_k^-)\bigr)$ and $\bigl(\delta_k^+, w_j(\delta_k^+)\bigr)$, on two opposite sides of the rectangle $[\delta_k^-, \delta_k^+] \times [\delta_j^-, \delta_j^+]$. Inequalities (19) are a byproduct of the proof of Lemma 10; only the inequality $\delta_j^- < w_j(\delta_k)$ deserves a comment. If $\delta \in \Sigma_j \cap \Pi$ and $\delta_k = \delta_k^-$ then we have $G_j(\delta) > G^-(\delta_j)$ as defined by (18), so that the inequality is satisfied by the solution of Equation (11). Q.E.D.

In view of Lemma 11, we may redefine the surfaces Σ_j as

$$\Sigma_j = \bigl\{\delta \in \mathbb{R}^n : \delta_j = w_j(\Delta_j)\bigr\}\ . \tag{20}$$

The solution of the system of Equations (11) should be determined by the intersection of the surfaces Σ_j. This is the issue to be dealt with.

2.2. *The case of three bodies*

Here, as a significant example, we consider the case $N = 3$, hence $n = 2$, which can be illustrated with the help of figures. This is indeed the case first studied by Euler and Lagrange, who, however, considered only the case of Newton's gravitational force.

Writing Equations (10) and (11) in explicit form we get the system

$$\begin{aligned}\Psi\delta_1 - (m_1 + m_2)\varphi(\delta_1) &= -m_3\bigl[\varphi(\delta_2) - \varphi(\delta_1 + \delta_2)\bigr]\ ,\\ \Psi\delta_2 - (m_2 + m_3)\varphi(\delta_2) &= -m_1\bigl[\varphi(\delta_1) - \varphi(\delta_1 + \delta_2)\bigr]\ .\end{aligned} \tag{21}$$

In view of Lemma 10 we know that a solution (δ_1^*, δ_2^*) of the equations, if any, must lie inside the rectangle $\Pi = (\delta_1^-, \delta_1^+) \times (\delta_2^-, \delta_2^+)$.

The equations are symmetric in δ_1 and δ_2; hence it is enough to discuss the second one, so that we have $j = 2$ and $\Delta_2 = \{\delta_1\}$. In Fig. 2 we draw the qualitative graph of the functions $w_1(\delta_2)$ (blue) and $w_2(\delta_1)$ (red) in different possible cases. The rectangle Π where the desired solution may be located is highlighted; grey regions are progressively excluded.

In view of Lemma 11 and Corollary 12, the second Equation (21) determines a curve Σ_2, defined by a continuous, one-valued and monotonically increasing function $w_2(\delta_1)$. The former property excludes the panel (a); the latter excludes the panel (b).

The graph of $w_2(\delta_1)$ for $\delta_1 \to 0$ may deserve some comment. This is not strictly necessary, for we are actually interested only on the interval $\delta_1 \geq \delta_1^-$,

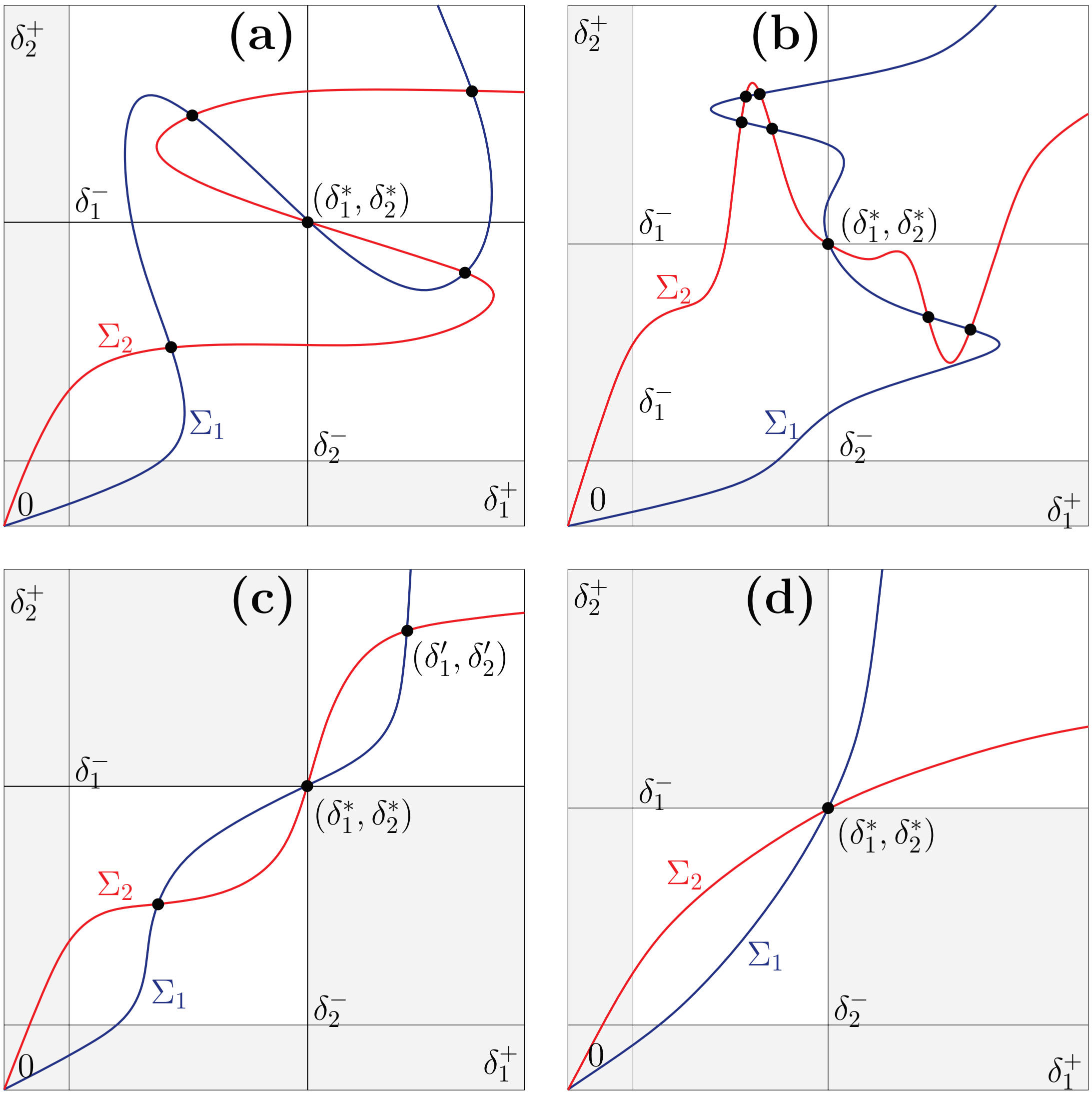

Fig. 2 – Schematic representation of the curves Σ_1, Σ_2 defined by Equations $Q_1(\delta_1) = G_1(\delta_1, \delta_2)$ and $Q_2(\delta_2) = G_2(\delta_1, \delta_2)$, inside the rectangle Π. The four panels suggest some possible layout of the graphs.

but it may help to clarify the qualitative behaviour of the curve close to the origin. The critical case shows up if we let $\varphi(0) = +\infty$; we are confronted with a double limit $\delta_1 \to 0$ and $\delta_2 \to 0$. We are actually interested in determining $\lim_{\delta_2 \to 0} w_1(\delta_2)$; hence we proceed as follows. Let us choose a small $\varepsilon > 0$, and redefine $\varphi(r) = \varphi(\varepsilon)$ for $r < \varepsilon$. For $\delta_1 \to 0$ we just replace the infinite value $\varphi(0)$ in Equation (21) with the finite $\varphi(\varepsilon)$. Thus we get a positive solution $\delta_1' = w_1(0)$, depending on ε. In the limit $\varepsilon \to 0$, for $\delta_1 < \varepsilon$

we keep only the dominating terms, thus rewriting Equation (21) as

$$\varphi(\delta_2) \simeq \frac{m_1}{m_1 + m_2 + m_3} \varphi(\delta_1) \ .$$

This shows that for small δ_1 we have $w_2(\delta_1) > \delta_1$, with $\delta_2 \to 0$, as represented in the figure. The same considerations, mutatis mutandis, apply to the first Equation (21), and so to the curve Σ_1 and to the function $w_1(\delta_2)$. We just note that when we let $\delta_2 \to 0$ we should exchange the role of the two limits, which makes the apparent intersection $\Sigma_1 \cap \Sigma_2$ at the origin in the figures a spurious fact.

Let us now focus on the intersection $\Sigma_1 \cap \Sigma_2$, looking at panels (c) and (d) of Fig. 2. We aim at proving that an intersection point exists, and that it is unique. From Corollary 12–(iii), we know that $\Sigma_1 \cap \Pi$ joins continuously two points on the lower and upper side of the rectangle Π; similarly, $\Sigma_2 \cap \Pi$ joins continuously two points on the left and right side of the rectangle. In view of Bolzano's theorem, the two curves must have an intersection point. It remains to show that this point is unique.

By contradiction, suppose that there exists $\delta' \in \Sigma_1 \cap \Sigma_2$ such that $\delta^* \neq \delta'$. By possibly exchanging the points δ^* and δ', we may consider only two different cases, namely: (i) $\delta'_1 \leq \delta^*_1$ and $\delta'_2 > \delta^*_2$; or (ii) $\delta'_1 > \delta^*_1$ and $\delta'_2 > \delta^*_2$. We show that neither of these two cases occurs. The case (i) is that of panels (a) or (b) of Fig. 2, that we have already excluded. In case (ii), both δ^* and δ' must satisfy also Equation (12) for the length $\ell = \delta_1 + \delta_2$, which in our case is written as

$$\Psi\ell - (m_1 + m_3)\varphi(\ell) = m_2\bigl[\varphi(\delta_1) + \varphi(\delta_2)\bigr] \ ,$$

with the corresponding lengths $\ell^* = \delta^*_1 + \delta^*_2$ and $\ell' = \delta'_1 + \delta'_2$. This cannot be, for, by increasing both δ_1 and δ_2, the left member increases, and the right member decreases, in view of the monotonically decreasing character of $\varphi(r)$. This excludes the case of panel (c), thus showing that panel (d), with a single intersection point, is the correct one.

As an example, consider the case of Newtonian force $\varphi(r) = r^{-2}$, using some algebra. We may conveniently introduce better variables, adapted to the case $n = 2$. Using the length $\ell = \xi_3 - \xi_1$ as a parameter, we set $\delta_1 = \sigma$ and $\delta_2 = \ell - \sigma$, and rewrite Equations (21) as

$$(22) \qquad \begin{aligned} \Psi\sigma - (m_1 + m_2)\varphi(\sigma) &= -m_3\bigl[\varphi(\ell - \sigma) - \varphi(\ell)\bigr] \ , \\ \Psi(\ell - \sigma) - (m_2 + m_3)\varphi(\ell - \sigma) &= -m_1\bigl[\varphi(\sigma) - \varphi(\ell)\bigr] \ . \end{aligned}$$

Multiplying the first equation by $(\ell-\sigma)$ and the second by σ, and subtracting member by member we get

$$\begin{aligned}(m_1+m_2)(\ell-\sigma)\varphi(\sigma) - m_3(\ell-\sigma)\bigl[\varphi(\ell-\sigma)-\varphi(\ell)\bigr] \\ = (m_2+m_3)\sigma\varphi(\ell-\sigma) - m_1\sigma\bigl[\varphi(\sigma)-\varphi(\ell)\bigr] \ .\end{aligned}$$

Now we introduce a variable $x = \sigma/\ell$, with $x \in (0,1)$, and set $\varphi(r) = \frac{1}{r^2}$. Rearranging the equation we write

$$\frac{m_1+m_2(1-x)}{x^2} - \frac{m_3+m_2x}{(1-x)^2} = m_1x - m_3(1-x) \ .$$

We are thus led to solve the algebraic equation of fifth degree

$$\begin{aligned}(m_3+m_1)\,x^5 - (3m_3+2m_1)\,x^4 + (3m_3+2m_2+m_1)\,x^3 \\ - (3m_2+m_1)\,x^2 + (3m_2+2m_1)\,x - m_1 - m_2 = 0 \ .\end{aligned}$$

We know from our general argument that a unique solution exists; on need, we calculate it through numerical methods. For given ℓ, this gives δ_1 and δ_2, while Ψ may be found through Equation (12). The discussion of the orbits goes as in the general case.

We have thus recovered the collinear solutions discovered by Euler (Euler 1767; Euler 1770) and Lagrange (Lagrange 1772). However, we emphasize once again that our theory applies also to the case of generic forces satisfying our hypotheses (H1)–(H3).

2.3. Proof of Proposition 8

Coming to the general case $N > 3$, we study the intersections of the surfaces Σ_j defined by Equation (20). Corollary 12 plays a main role here. We use notations such as $F(\delta)\big|_{\Sigma_j}$ to denote the restriction of a function $F(\delta)$ to the surface Σ_j. That is, $F(\delta)\big|_{\Sigma_j} = F(\delta)\big|_{\delta_j=w_j(\Delta_j)}$, so that the function on the left side is made independent of δ_j, and depends only on $n-1$ variables.

We construct the proof through a recurrent procedure, schematically represented in Fig. 3. Let us define the sets

$$(23) \qquad \Omega_1 = \Sigma_1 \ , \quad \Omega_s = \Omega_{s-1} \cap \Sigma_s \quad \text{for} \quad s = 2,\ldots,n \ .$$

Our aim is to prove that $\{\Omega_s\}$ is a sequence of surfaces of decreasing dimension $n-s$, ending up with a single point δ^* which is the solution of

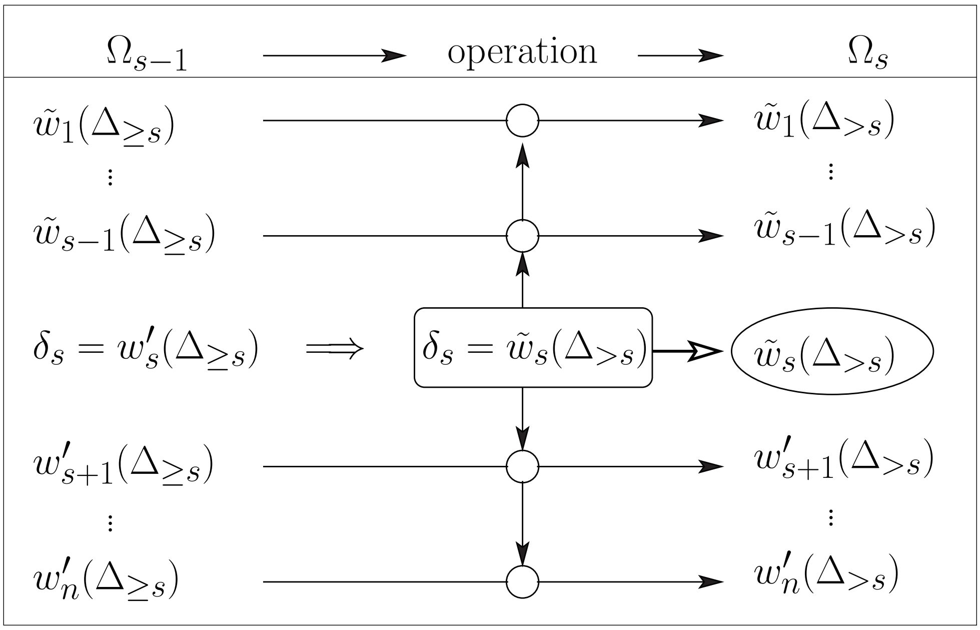

Fig. 3 – Schematic representation of the recurrent proof of Proposition 8

Equations (10). We use the simplified notation $\Delta_{<s} = \{\delta_1, \ldots, \delta_{s-1}\}$, and similar notations $\Delta_{\leq s}$, $\Delta_{>s}$ and $\Delta_{\geq s}$, with obvious meaning.

At every step $s = 1, \ldots, n-1$, we construct by recurrence a set of functions

$$\begin{aligned} &\tilde{w}_j(\Delta_{>s}) \quad \text{for} \quad j = 1, \ldots, s\ , \\ &w'_k(\Delta_{>s}) \quad \text{for} \quad k = s-1, \ldots, n\ ; \end{aligned}$$

(we avoid overloading the notations by introducing a further label s which identifies the step; the scheme shows how the functions actually change, and the functions on the right of the scheme completely replace those on the left). The $(n-s)$–dimensional surface Ω_s (in the right column of Fig. 3) is described by the s functions $\tilde{w}_1(\Delta_{>s}), \ldots, \tilde{w}_s(\Delta_{>s})$. The remaining functions $w'_k(\Delta_{>s})$ are defined as $w'_k(\Delta_{>s}) = w_k(\Delta_k)\big|_{\Omega_s}$, namely the restriction of the initial functions to the surface Ω_s. A key point is that at every step the functions $w'_k(\Delta_{>s})$ enjoy the properties of Corollary 12, that is, they are continuous and monotonically increasing, and satisfy the inequalities (19). Now we describe the procedure in algorithmic form, postponing the proof of these properties.

The first step, $s = 1$, is a trivial matter. Noting that $\Delta_{>1} = \Delta_1$, we fill the left column of the scheme in Fig. 3, corresponding to $s - 1 = 1$, by just setting

$$\begin{aligned} \delta_1 &= \tilde{w}_1(\Delta_{>1}) = w_1(\Delta_1)\ ; \\ \delta_k &= w'_k(\Delta_{>1}) = w_k(\Delta_k)\big|_{\delta_1 = \tilde{w}_1(\Delta_{>1})}\ , \quad k = 2, \ldots, n\ . \end{aligned}$$

The substitution $\delta_1 = \tilde{w}_1(\Delta_{>1})$ on the last line performs the restriction $w_k(\Delta_k)\big|_{\Omega_1}$; the variable δ_1 is made to depend on $\Delta_{>1}$, which reduces the dimension. The point to be highlighted is that $\delta_2 = w_s(\Delta_2)$ defines the surface Σ_2, since the second member is independent of δ_2; the substitution removes δ_1 from the free arguments, but reintroduces δ_2, thus promoting the definition to be an equation for δ_2.

Now we come to the recurrent part of the proof, $s = 2, \ldots, n-1$. Suppose that we have filled the left column of the scheme for Ω_{s-1}. We focus on the central row of the scheme. Solving the equation $\delta_s = w'_s(\Delta_{>s})$ with respect to δ_s, we get the new function $\delta_s = \tilde{w}_s(\Delta_{>s})$, written in the central rectangle and replicated in the right column; this replaces the function $w'_s(\Delta_{\geq s})$ on the left. The other lines mean that in each function on the left, either $\tilde{w}_j(\Delta_{\geq s})$ or $w'_k(\Delta_{\geq s})$, we substitute $\delta_s = \tilde{w}_s(\Delta_{>s})$, symbolized by the small circles; this represents indeed the restriction, either $\tilde{w}_j(\Delta_{\geq s})\big|_{\Omega_s}$ or $w'_k(\Delta_{\geq s})\big|_{\Omega_s}$, reported in the right column as depending only on $\Delta_{>s}$.

For $s = n$, in the left column we have functions of the sole variable δ_n, The solution of the equation $\delta_n = w'(\delta_n)$ provides a value satisfying $\delta^*_n = \tilde{w}_n(\delta^*_n)$. The other lines determine $\delta^*_j = \tilde{w}_j(\delta^*_n)$. This provides the final value $\delta^* = (\delta^*_1, \ldots, \delta^*_n)$ which is the desired solution of Equation (10).

This describes the algorithm. Here from, in view of Lemma 10, we restrict our attention to the rectangle Π. We should prove the following:

(i) that the solution of the equation $\delta_s = w'_s(\Delta_{>s})$ exists and is unique;

(ii) that the functions $\tilde{w}_j(\Delta_{>s})$ and $w'_k(\Delta_{>s})$ found at step s are continuous and monotonically increasing, and enjoy the properties of Corollary 12 with respect to the variables $\Delta_{>s}$ on which they depend.

It may seem natural to argue that a solution of the equation for δ_s should exist, and that it inherits the properties of Corollary 12, for we are selecting a restricted set among the solutions of the equation $Q_s(\delta_s) = G_s(\Delta_s)$, which do exist, indeed. This, however, may be a puzzling point; so let us clarify it. We make reference to Fig. 4.

For $s = 1$ we have $\tilde{w}_1(\Delta_{>1}) = w_1(\Delta_1)$, and there is nothing to prove. The functions $w'_k(\Delta_{>1}) = w_k(\Delta_k)\big|_{\Omega_1}$ are compositions of continuous and monotonically increasing functions, that only make δ_1 to depend on $\Delta_{>s}$. Hence, $w'_k(\Delta_{>1})$ inherits all properties, including those of Corollary 12.

Suppose that the claims (i) and (ii) are true up to $s-1$, and focus attention on the equation $\delta_s = w'_s(\Delta_{\geq s})$. Let, for a moment, $\Delta_{>s}$ be fixed; thus the equation means that we may restrict our attention to considering a continuous, monotonically increasing map $w'_s(\delta_s; \Delta_{>s})$, depending on the variables $\Delta_{>s}$ as parameters. By Corollary 12 the image of the map is an interval

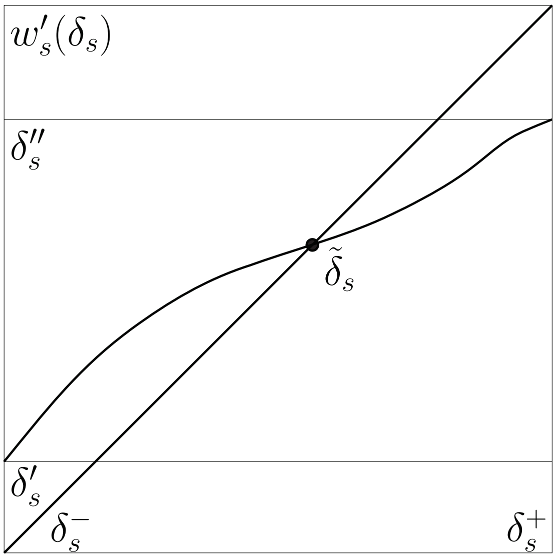

Fig. 4 – Illustrating the solution of the equation $\delta_s = w'_s(\Delta_{>s})$, for $\Delta_{>s}$ fixed, in the square $(\delta_s^-, \delta_s^+) \times (\delta_s^-, \delta_s^+)$.

$[\delta'_s, \delta''_s] \subset [\delta_s^-, \delta_s^+]$, with the properties stated in Equations (19). The qualitative graph of the map is represented by the curve in Fig. 4, to be intersected with the graph of the identity. By Bolzano's theorem, there is at least one fixed point $\tilde{\delta}_s$. We prove that such a point is unique, using Equation (12) for the length $\ell = \delta_1 + \ldots + \delta_n$ of the chain. By contradiction, suppose that there is a second fixed point $\overline{\delta}_s > \tilde{\delta}_s$. Since $\Delta_{>s}$ are fixed and all functions $\tilde{w}_j$ are monotonically increasing with δ_s, the left member $\hat{Q}(\ell)$ must increase. Conversely, the right member $\hat{G}(\delta)$, as defined by Equation (13), must decrease. Therefore a second fixed point $\overline{\delta}_s$ cannot exist.

Let now Δ_s to vary. In all previous steps we have just made a sequence of restrictions of the initial functions $w_s(\Delta_s)$ to the surfaces $\Omega_1, \ldots, \Omega_{s-1}$; thus we have $w'_s(\Delta_{\geq s}) = w_s(\Delta_s)\big|_{\Omega_{s-1}}$. That is, at every point $\Delta_{\geq s}$ we calculate the composed function

$$w'(\Delta_{\geq s}) = w_s\bigl(\tilde{w}_1(\Delta_{\geq s}), \ldots, \tilde{w}_{s-1}(\Delta_{\geq s}), \delta_s, \ldots, \delta_n\bigr) ,$$

Letting any of the variables $\Delta_{>s}$ to increase, and by the induction hypothesis, the functions $\tilde{w}_j(\Delta_{\geq s})$ increase too, and are continuous and one valued. By composition, $w'(\Delta_{\geq s})$ inherits all these properties. For any $\Delta_{\geq s}$ we determine an unique point $\tilde{\delta}_s(\Delta_{>s})$, thus constructing a function which is still continuous, one valued and monotonically increasing with every variable in $\Delta_{>s}$ (Fig. 4 may help). Furthermore, for any $\Delta_{\geq s}$ the interval $[\delta_s^-, \delta_s^+]$ is mapped into $[\delta'_s, \delta''_s] \subset [\delta_s^-, \delta_s^+]$, thus making $\tilde{\delta}_s(\Delta_{>s})$ to inherit the properties of Corollary 12. Let us define the new function

$$\tilde{w}_s(\Delta_{>s}) = w'_s(\Delta_{\geq s})\Big|_{\delta_s=\tilde{\delta}_s} . \tag{24}$$

This is again a composition of continuous, one valued and monotonically increasing functions, which for any $\Delta_{>s}$ maps the interval $[\delta_s^-, \delta_s^+]$ into itself, and retains all desired properties, including those of Corollary 12. This completes the induction step up to $n-1$.

For $s = n$ we use only the proof that the equation $\delta_n = w'(\delta_n)$ has a unique solution δ_n^*; thus the substitution $\delta_n = \delta_n^*$ makes the right column of the scheme in Fig. 3 to contain only unique numbers δ_j^*. As already said, this provides the final value $\delta^* = (\delta_1^*, \ldots, \delta_n^*)$ which is the desired solution of Equation (10). The proof of Proposition 8 is thus complete.

3. Relative equilibria and stationary orbits

We come to the proof of Theorem 1. We show that to every solution δ^* of the system of equations (11) there corresponds a relative equilibrium. The orbits are written as in (3), with the bodies aligned on a straight line at positions $\xi_1, \ldots, \xi_N$. These positions are calculated from $\delta_1, \ldots, \delta_n$, as follows. We first set, by recurrence, $\xi_1 = 0$ and $\xi_j = \xi_{j-1} + \delta_j$ for $j = 2, \ldots, N$; then we translate the origin to the barycentre. Equation (12) provides the length ℓ of the chain. According to Corollary 7, for a stationary orbit we have $\varrho(t) = \varrho_0$ and $\dot{\theta} = \omega = L/\varrho_0^2$, where L is determined from the relation $L^2/\varrho_0^3 = \Psi$. Recall that at the beginning of Section 4 we have introduced a rescaling so that $\varrho_0 = 1$. Thus, by Proposition 8, the desired stationary orbit is described as

$$\varrho(t) = 1 , \quad \omega = \sqrt{\Psi} .$$

Since a solution exists for any positive Ψ, we have actually found a whole one-parameter family of stationary orbits. Any such solution corresponds to a given order of the masses, which is given in advance. Hence, by applying an arbitrary permutation, and identifying symmetric solutions obtained by inversion of the order, we may find $N!/2$ different configurations. This completes the proof of Theorem 1.

4. Homographic orbits and homothetic orbits

The question concerning homographic orbits and homothetic orbits is more challenging: the point that deserves to be examined is whether a rescaling of coordinates with a factor $\varrho(t)$ may be introduced in a form compatible with Equations (4). As stated in Corollary 7, we should replace the constant Ψ Equation (3) with a function $\psi(\varrho)$, satisfying a compatibility relation with the function $\varphi(\varrho r)$.

Corollary 13. *Let us suppose that $\varphi(\varrho r) = g(\varrho)\varphi(r)$. Then, to any solution of Equations (8) with a given positive Ψ we may associate a family of solutions parameterized by $\varrho > 0$, just substituting the constant Ψ with the function $\psi(\varrho) = \Psi g(\varrho)$, and leaving $\xi = (\xi_1, \ldots, \xi_N)$ fixed.*

The proof is just matter of factoring out $g(\varrho)$ in the right members of Equations (8).

According to Corollary 7, the solution of Newton's equation for the central motion provides both functions $\varrho(t)$ and $\theta(t)$, the constant L being calculated from the initial data. Since all positions of the bodies are multiplied by a common factor $\varrho(t)$, the ratio between mutual distances of bodies remains constant.

Homothetic orbits are a particular case, when the initial conditions are such that $L = 0$. In this case the motion is rectilinear.

It is quite natural, though not necessary for a dynamical model, to assume that the force function $\varphi(r)$ should be differentiable. In this case we prove that the function should be homogeneous, hence a power law.

Lemma 14. *Let the force function $\varphi(r)$ be differentiable. Then we have $\varphi(r) \propto r^\alpha$ with some α, so that $\psi(\varrho) \propto \varrho^\alpha$.*

Proof. Let $\varrho = 1 + \varepsilon$, and write

$$\begin{aligned}\varphi\bigl((1+\varepsilon)r\bigr) &= \varphi(r) + \varepsilon r\varphi'(r) + o(\varepsilon) \ , \\ g(1+\varepsilon)\varphi(r) &= \bigl[g(1) + \varepsilon g'(1) + o(\varepsilon)\bigr]\varphi(r) \ .\end{aligned}$$

Setting $g(1) = 1$ and $g'(1) = \alpha$ and equating the coefficients of ε we immediately get the equation $r\varphi'(r) = \alpha\varphi(r)$. By separation of variables we get $\varphi(r) = r^\alpha$ up to a multiplicative factor, as claimed. Q.E.D.

This Lemma justifies the introduction of our hypothesis (H4). The proof of Theorem 2 is thus concluded.

We illustrate our result by drawing some orbits for three bodies in Fig. 5; the bodies are labeled as A (green), B (red) and C (blue). In the upper left panel we draw the orbits for the Newtonian gravitational force, $\varphi(r) = 1/r^2$: the orbits are homographic ellipses. The right upper plane shows the orbits for the force $\varphi(r) = 1/r^{3/2}$; the orbits are still homographic, and exhibit

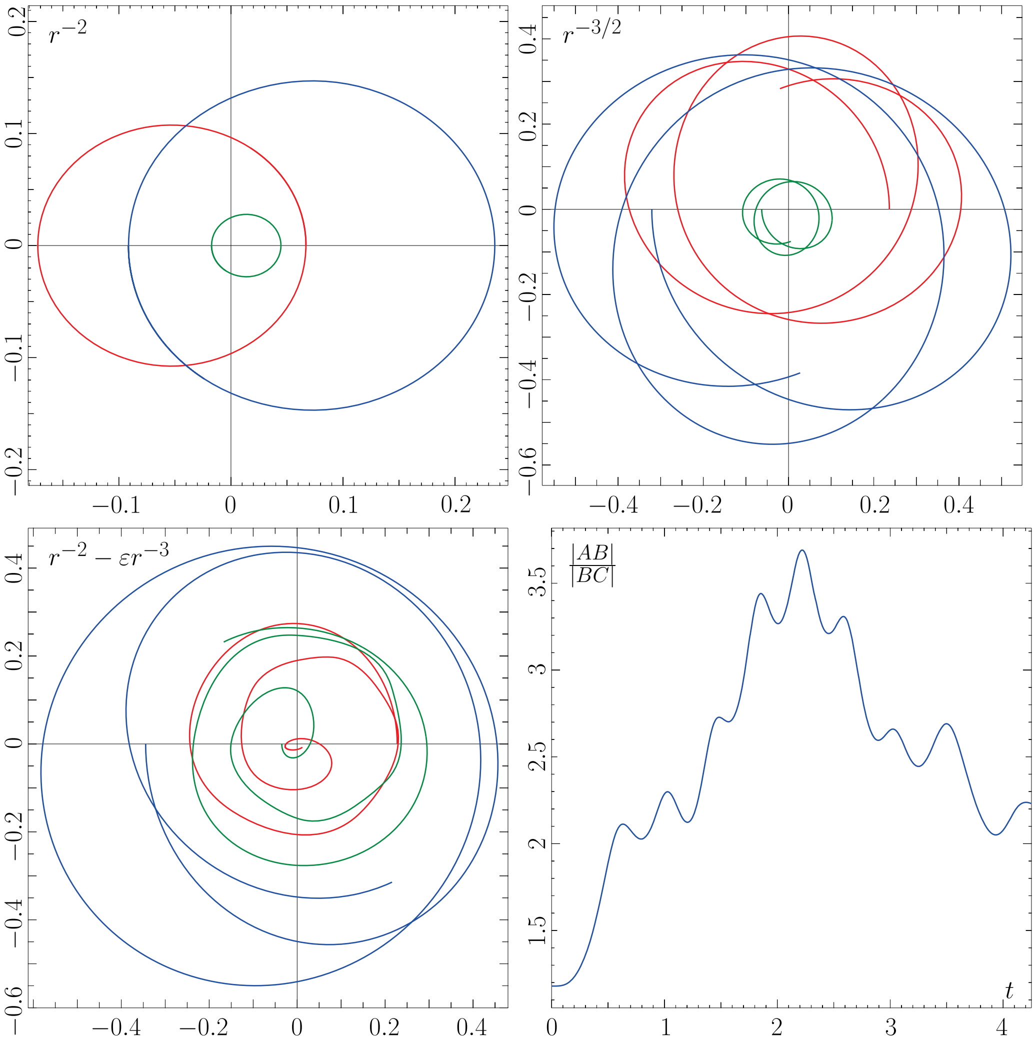

Fig. 5 – Collinear solutions of the problem of three bodies. Upper left panel: $\varphi(r) = 1/r^2$, homographic ellipses. Upper right panel: $\varphi(r) = 1/r^{3/2}$, homographic orbits with apsidal precession. Lower left panel: a non-homogeneous force $\varphi(r) = 1/r^2 - \varepsilon/r^3$, with $\varepsilon = 0.225$; the orbits are not homographic. Lower right panel: the time evolution of the ratio $|AB| / |BC|$.

an apsidal precession with an angle independent of the radius, as predicted by Newton (see, for instance, (Giorgilli and Guicciardini 2021). The lower panels refer to the non-homogeneous force $\varphi(r) = 1/r^2 - \varepsilon/r^3$, where ε is a positive parameter; stationary (circular) orbits do exist as well, but non-stationary orbits are not homographic, as predicted by our theory. This is confirmed by the lower right panel: the time evolution of the ratio of the distances $|AB| / |BC|$ is not constant.

5. Conclusions

We have proved the existence of collinear central configurations for a system of N bodies, thus generalizing the solutions of the problem of three bodies found by Euler and Lagrange. With respect to the existing literature, we consider a general class of forces, obeying mild conditions of monotonicity, continuity and convexity. This includes in particular, as a subclass, forces depending on an inverse power of distance.

As a general fact, we prove the existence of stationary solutions, akin to relative equilibria, with the bodies revolving around the barycentre in a rigid configuration. Then we prove the existence of homographic and homothetic orbits by restricting the forces to obey an inverse power law. The latter forces are widely considered in Newton's *Principia.*

Our discussion reduces the study concerning the qualitative shape of homographic orbits to that of a single body moving in a central force field. A detailed discussion of the latter point would require a considerable number of pages, which is incompatible with the length of an article like this. On the other hand, it may be found on classical treatises on Mechanics or Celestial Mechanics — Newton's, for instance.

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References

Euler L. (1766). *Considerationes de motu corporum coelestium*. In: «Novi Commentarii Academiae Scientiarum Petropolitanae», 10: 544-558 [Reprinted in: Id. (1952), *Opera omnia*, s. II, vol. 25, 246-257].

Euler L. (1767). *De motu rectilineo trium corporum se mutuo attrahentium*. In: «Novi Commentarii Academiae Scientiarum Petropolitanae», 11: 144-151 [Reprinted in: Id. (1952), *Opera omnia*, s. II, vol. 25, 281-289].

Euler L. (1770). *Considérations sur le problème des trois corps*. In: «Mémoires de l'Académie des Sciences de Berlin», 19: 194-220 [Reprinted in: Id. (1953), *Opera omnia*, s. II, vol. 26, 246-257].

Giorgilli A. and N. Guicciardini (2021). *La legge gravitazionale dell'inverso del quadrato nei* Principia *di Newton*. In: «Rendiconti dell'Istituto Lombardo Accademia di Scienze e Lettere – Classe di Scienze matematiche e naturali», 155: 169-196.

Lagrange J.L. (1772). *Essai sur le problème des trois corps*. In: «Mémoires de l'Académie des Sciences de Paris» [Reprinted in: Id. (1873), *Oeuvres de Lagrange*, VI t. Paris: Gauthier-Villars, 229-331].

Lehmann-Filhés R. (1891). *Ueber zwei Fälle des Vielkörperproblems*. In: «Astronomische Nachrichten», 127: 137-144.

Moeckel R. (1914). *Lectures on central con gurations*. In: *Lecture notes of an advanced course at the Centre de Recerca Matemàtica*, Barcelona. Available at the url: *www-users.cse.umn.edu/~rmoeckel/notes/Notes.html* [last access September 18, 2025].

Moulton F.R. (1910). *The straight line solutions of the problem of* N *bodies*. In: «Annals of Mathematics», s. 2, 12: 1-17.

Newton I. (1726). *Philosophiæ Naturalis Principia Mathematica*, *auctore Is. Newton, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali, Londini, jussu Societatis Regiæ ac typis Josephi Streater, Anno MDCLXXXVII*. London: Guil & Joh. Innys, Regiæ Societatis typographos [third edition].

Saari D.G. (2005). *Collisions, rings, and other Newtonian* N-*body problems*. In: *CBMS Regional conference series in mathematics*, vol. 104. Providence: American Mathematical Society.

Id. (2011). *Central configurations – A problem for the twenty-first century*. In: Shubin, T., Hayes, D., Alexanderson, G., eds., *Expeditions in mathematics*, MAA Spectrum. Washington, DC: Mathematical Association of America, 283-297.

Wintner A. (1941). *The analytical foundations of celestial mechanics*. Princeton: University Press.